Title	On the structure of multiple-scale solutions of the Painleve equations with a large parameter(Algebraic Analysis of Singular Perturbations)
Author(s)	KAWAI, Takahiro; TAKEI, Yoshitsugu
Citation	数理解析研究所講究録 (1996), 968: 49-59
Issue Date	1996-10
URL	http://hdl.handle.net/2433/60641
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
968 巻 1996 年 49-59

On the structure of multiple-scale solutions of the Painlevé equations with a large parameter

Takahiro KAWAI (京大数理研 河合隆裕)
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-01 JAPAN

and

Yoshitsugu TAKEI (京大数理研 竹井義次)
Research Institute for Mathematical Sciences
Kyoto University
Kyoto, 606-01 JAPAN

§0. Introduction.

The purpose of this talk is to report that our conjecture on multiple-scale solutions of Painlevé equations (P_J) $(J = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VI}$; cf. Table 0.1 below) with a large parameter η has been proved; each 2-parameter multiple-scale solution of (P_J) is locally reduced to a suitably chosen 2-parameter multiple-scale solution of the first Painlevé equation (P_{I}). (See Theorem 2.1 for the precise statement.)

This is a natural generalization of the result on 0-parameter solutions. ([KT1, Theorem 2.3.])

The details of this report will appear in [KT3].

Although we use the same notations as in [AKT], we list up basic equations and related symbols for the sake of definiteness. In what follows, J ranges over $\{\mathrm{I}, \mathrm{II}, \cdots, \mathrm{VI}\}$ unless otherwise stated.

Table 0.1. *Painlevé equations with a large parameter* η.

(P_{I}) $$\frac{d^2\lambda}{dt^2} = \eta^2(6\lambda^2 + t).$$

(P_{II}) $$\frac{d^2\lambda}{dt^2} = \eta^2(2\lambda^3 + t\lambda + \alpha).$$

(P_{III}) $$\frac{d^2\lambda}{dt^2} = \frac{1}{\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + 8\eta^2\left[2\alpha_\infty\lambda^3 + \frac{\alpha'_\infty}{t}\lambda^2 - \frac{\alpha'_0}{t} - 2\frac{\alpha_0}{\lambda}\right].$$

(P_{IV}) $$\frac{d^2\lambda}{dt^2} = \frac{1}{2\lambda}\left(\frac{d\lambda}{dt}\right)^2 - \frac{2}{\lambda} + 2\eta^2\left[\frac{3}{4}\lambda^3 + 2t\lambda^2 + (t^2 + 4\alpha_1)\lambda - \frac{4\alpha_0}{\lambda}\right].$$

(P_{V}) $$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \left(\frac{1}{2\lambda} + \frac{1}{\lambda-1}\right)\left(\frac{d\lambda}{dt}\right)^2 - \frac{1}{t}\frac{d\lambda}{dt} + \frac{(\lambda-1)^2}{t^2}\left(2\lambda - \frac{1}{2\lambda}\right)\\ &\quad + \eta^2\frac{2\lambda(\lambda-1)^2}{t^2}\left[(\alpha_0 + \alpha_\infty) - \alpha_0\frac{1}{\lambda^2}\right.\\ &\quad \left. - \alpha_2\frac{t}{(\lambda-1)^2} - \alpha_1 t^2\frac{\lambda+1}{(\lambda-1)^3}\right].\end{aligned}$$

(P_{VI}) $$\begin{aligned}\frac{d^2\lambda}{dt^2} &= \frac{1}{2}\left(\frac{1}{\lambda} + \frac{1}{\lambda-1} + \frac{1}{\lambda-t}\right)\left(\frac{d\lambda}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{\lambda-t}\right)\frac{d\lambda}{dt}\\ &\quad + \frac{2\lambda(\lambda-1)(\lambda-t)}{t^2(t-1)^2}\left[1 - \frac{\lambda^2 - 2t\lambda + t}{4\lambda^2(\lambda-1)^2}\right.\\ &\quad + \eta^2\left\{(\alpha_0 + \alpha_1 + \alpha_t + \alpha_\infty) - \alpha_0\frac{t}{\lambda^2}\right.\\ &\quad \left.\left. + \alpha_1\frac{t-1}{(\lambda-1)^2} - \alpha_t\frac{t(t-1)}{(\lambda-t)^2}\right\}\right].\end{aligned}$$

Table 0.2. *Painlevé Hamiltonian systems with a large parameter η.*

$$(H_J) \qquad \begin{cases} \dfrac{d\lambda}{dt} = \eta \dfrac{\partial K_J}{\partial \nu} \\[2mm] \dfrac{d\nu}{dt} = -\eta \dfrac{\partial K_J}{\partial \lambda} \end{cases},$$

where K_J is tabulated below:

$$K_{\mathrm{I}} = \frac{1}{2}\left[\nu^2 - (4\lambda^3 + 2t\lambda)\right].$$

$$K_{\mathrm{II}} = \frac{1}{2}\left[\nu^2 - (\lambda^4 + t\lambda^2 + 2\alpha\lambda)\right].$$

$$K_{\mathrm{III}} = \frac{2\lambda^2}{t}\left[\nu^2 - \eta^{-1}\frac{3\nu}{2\lambda} - \left(\frac{\alpha_0 t^2}{\lambda^4} + \frac{\alpha_0' t}{\lambda^3} + \frac{\alpha_\infty' t}{\lambda} + \alpha_\infty t^2\right)\right].$$

$$K_{\mathrm{IV}} = 2\lambda\left[\nu^2 - \eta^{-1}\frac{\nu}{\lambda} - \left(\frac{\alpha_0}{\lambda^2} + \alpha_1 + \left(\frac{\lambda + 2t}{4}\right)^2\right)\right].$$

$$\begin{aligned} K_{\mathrm{V}} = {} & \frac{\lambda(\lambda-1)^2}{t} \\ & \times \left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\nu \right. \\ & \left. - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1 t^2}{(\lambda-1)^4} + \frac{\alpha_2 t}{(\lambda-1)^3} + \frac{\alpha_\infty}{(\lambda-1)^2}\right)\right]. \end{aligned}$$

$$\begin{aligned} K_{\mathrm{VI}} = {} & \frac{\lambda(\lambda-1)(\lambda-t)}{t(t-1)} \\ & \times \left[\nu^2 - \eta^{-1}\left(\frac{1}{\lambda} + \frac{1}{\lambda - 1}\right)\nu \right. \\ & \left. - \left(\frac{\alpha_0}{\lambda^2} + \frac{\alpha_1}{(\lambda-1)^2} + \frac{\alpha_\infty}{\lambda(\lambda-1)} + \frac{\alpha_t}{(\lambda-t)^2}\right)\right]. \end{aligned}$$

Table 0.3. *Relevant Schrödinger equations with a large parameter η.*

$$(SL_J) \qquad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_J(x,t,\eta)\right)\psi_J(x,t,\eta) = 0,$$

where Q_J is given below. (See the preceding Table 0.2 for the symbol K_J used there.)

$$Q_{\mathrm{I}} = 4x^3 + 2tx + 2K_{\mathrm{I}} - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2}.$$

$$Q_{\mathrm{II}} = x^4 + tx^2 + 2\alpha x + 2K_{\mathrm{II}} - \eta^{-1}\frac{\nu}{x-\lambda} + \eta^{-2}\frac{3}{4(x-\lambda)^2}.$$

$$\begin{aligned} Q_{\mathrm{III}} = {} & \frac{\alpha_0 t^2}{x^4} + \frac{\alpha_0' t}{x^3} + \frac{\alpha_\infty' t}{x} + \alpha_\infty t^2 + \frac{tK_{\mathrm{III}}}{2x^2} \\ & + \eta^{-1}\left(\frac{1}{2x^2} - \frac{1}{x(x-\lambda)}\right)\lambda\nu + \eta^{-2}\frac{3}{4(x-\lambda)^2}. \end{aligned}$$

$$Q_{\mathrm{IV}} = \frac{\alpha_0}{x^2} + \alpha_1 + \left(\frac{x+2t}{4}\right)^2 + \frac{K_{\mathrm{IV}}}{2x} - \eta^{-1}\frac{\lambda\nu}{x(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}.$$

$$\begin{aligned} Q_{\mathrm{V}} = {} & \frac{\alpha_0}{x^2} + \frac{\alpha_1 t^2}{(x-1)^4} + \frac{\alpha_2 t}{(x-1)^3} + \frac{\alpha_\infty}{(x-1)^2} + \frac{tK_{\mathrm{V}}}{x(x-1)^2} \\ & - \eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}. \end{aligned}$$

$$\begin{aligned} Q_{\mathrm{VI}} = {} & \frac{\alpha_0}{x^2} + \frac{\alpha_1}{(x-1)^2} + \frac{\alpha_\infty}{x(x-1)} + \frac{\alpha_t}{(x-t)^2} + \frac{t(t-1)K_{\mathrm{VI}}}{x(x-1)(x-t)} \\ & - \eta^{-1}\frac{\lambda(\lambda-1)\nu}{x(x-1)(x-\lambda)} + \eta^{-2}\frac{3}{4(x-\lambda)^2}. \end{aligned}$$

Table 0.4. *Deformation equations.*

(D_J)
$$\frac{\partial\psi}{\partial t} = A_J\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial A_J}{\partial x}\psi,$$

where A_J denotes the function given below:

$$A_{\mathrm{I}} = A_{\mathrm{II}} = \frac{1}{2(x-\lambda)}.$$

$$A_{\mathrm{III}} = \frac{2\lambda x}{t(x-\lambda)} + \frac{x}{t}.$$

$$A_{\mathrm{IV}} = \frac{2x}{x-\lambda}.$$
$$A_{\mathrm{V}} = \frac{\lambda-1}{t}\frac{x(x-1)}{x-\lambda}.$$
$$A_{\mathrm{VI}} = \frac{\lambda-t}{t(t-1)}\frac{x(x-1)}{x-\lambda}.$$

We note that (SL_J) and (D_J) are in involution (i.e., compatible) if (λ,ν) obeys the Hamiltonian system (H_J), which is known to be equivalent to (P_J).

§1. A canonical form of (SL_J) and (D_J) near the double turning point.

Let us consider the following pair of equations (Can) and (D_{can}), where ρ and σ are functions of t and η:

$$(Can) \qquad \left(-\frac{\partial^2}{\partial x^2} + \eta^2 Q_{\mathrm{can}}(x,\rho,\sigma,\eta)\right)\psi = 0$$

with

$$(1.1) \qquad Q_{\mathrm{can}} = 4x^2 + \eta^{-1}E + \frac{\eta^{-3/2}\rho}{x-\eta^{-1/2}\sigma} + \frac{3\eta^{-2}}{4(x-\eta^{-1/2}\sigma)^2},$$

where $E = \rho^2 - 4\sigma^2$,

and

$$(D_{\mathrm{can}}) \qquad \frac{\partial\psi}{\partial t} = A_{\mathrm{can}}\frac{\partial\psi}{\partial x} - \frac{1}{2}\frac{\partial A_{\mathrm{can}}}{\partial x}\psi,$$

where

$$(1.2) \qquad A_{\mathrm{can}} = \frac{1}{2(x-\eta^{-1/2}\sigma)}.$$

One can readily verify that equations (Can) and (D_{can}) are in involution if ρ and σ satisfy the following equation:

$$(H_{\mathrm{can}}) \qquad \begin{cases} \dfrac{d\rho}{dt} = -4\eta\sigma \\ \dfrac{d\sigma}{dt} = -\eta\rho \end{cases}.$$

For the sake of clarity of notations, we use the symbol $(\rho_{\text{can}}, \sigma_{\text{can}})$ to denote a solution of (H_{can}); note that ρ_{can} and σ_{can} are hyperbolic functions.

As is shown in Proposition 1.3 of [KT1], the top term $\lambda_0(t)$ of a multiple-scale solution of (P_J) gives rise to a double turning point of (SL_J). An important fact proved in Theorem 1.1 of [KT1] is that S_{odd}, the odd part of a solution S of the Riccati equation associated with (SL_J), is holomorphic near $x = \lambda_0(t)$ as far as we are concerned with 0-parameter solutions. Furthermore this regularity result leads to a very simple canonical form of the equation near the double turning point. (Cf. Theorem 1.2 of [KT1].) Although such a clear-cut result cannot be expected for 2-parameter multiple-scale solutions, we can still confirm the following Proposition 1.1 concerning the structure of simultaneous equations (SL_J) and (D_J) near the double turning point $x = \lambda_0(t)$. The relation (1.8.a) below is the counterpart of the canonical form for 0-parameter solutions, and the proposition plays a crucially important role in the proof of Theorem 2.1. For the sake of clarity of presentation we put $\sim$ over the variables and functions relevant to (SL_J), like $\tilde{x}, \tilde{t}$, etc., in the proposition. We also fix a point $\tilde{t}_*$ at a generic point as in Proposition 1.1 of [KT1], and we choose and fix sufficiently small neighborhoods $\widetilde{U}$ and $\widetilde{V}$ of $\tilde{x} = \tilde{\lambda}_0(\tilde{t}_*)$ and $\tilde{t}_*$, respectively.

Proposition 1.1. *For each $J = \mathrm{I}, \mathrm{II}, \cdots, \mathrm{VI}$, there exist holomorphic functions $x_{j/2}(\tilde{x}, \tilde{t}, \eta)$ and $t_{j/2}(\tilde{t}, \eta)$ $(j = 0, 1, 2, \cdots; (\tilde{x}, \tilde{t}) \in \widetilde{U} \times \widetilde{V})$ which satisfy the following relations:*

(1.3) $\quad x_0$ *and* t_0 *are independent of* η,

(1.4) $\quad \dfrac{\partial x_0}{\partial \tilde{x}}$ *never vanishes on* $\widetilde{U} \times \widetilde{V}$,

(1.5) $\quad x_0(\tilde{\lambda}_0(\tilde{t}), \tilde{t}) = 0$ *holds on* $\widetilde{V}$,

(1.6) $\quad t_0(\tilde{t}) = \tilde{\phi}_J(\tilde{t})/2$ *holds on* $\widetilde{V}$, *where*

$$\tilde{\phi}_J(\tilde{t}) = \int_{\tilde{r}}^{\tilde{t}} \sqrt{\frac{\partial \widetilde{F}_J}{\partial \tilde{\lambda}}(\tilde{\lambda}_0(\tilde{s}), \tilde{s})}d\tilde{s}$$

with $\tilde{r}$ being a turning point for $\tilde{\lambda}_J^{(0)}$ and with $\widetilde{F}_J$ denoting the coefficient of η^2 in the equation (P_J),

(1.7) $x_{1/2}(\tilde{x}, \tilde{t}, \eta)$ *and* $t_{1/2}(\tilde{t}, \eta)$ *identically vanish,*

(1.8) *If we set* $x(\tilde{x}, \tilde{t}, \eta) = \sum_{j\geq 0} x_{j/2}(\tilde{x}, \tilde{t}, \eta)\eta^{-j/2}$ *and* $t(\tilde{t}, \eta) = \sum_{j\geq 0} t_{j/2}(\tilde{t}, \eta)\eta^{-j/2}$, *then*

(1.8.a)
$$\widetilde{Q}_J(\tilde{x}, \tilde{t}, \eta) = \left(\frac{\partial x}{\partial \tilde{x}}\right)^2 Q_{\mathrm{can}}(x(\tilde{x}, \tilde{t}, \eta), \rho_{\mathrm{can}}(t(\tilde{t}, \eta), \eta), \sigma_{\mathrm{can}}(t(\tilde{t}, \eta), \eta), \eta) - \frac{1}{2}\eta^{-2}\{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\},$$

where
$$\{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\} = \frac{\dfrac{\partial^3 x}{\partial \tilde{x}^3}}{\dfrac{\partial x}{\partial \tilde{x}}} - \frac{3}{2}\left(\frac{\dfrac{\partial^2 x}{\partial \tilde{x}^2}}{\dfrac{\partial x}{\partial \tilde{x}}}\right)^2,$$

(1.8.b)
$$\sigma_{\mathrm{can}}(t(\tilde{t}, \eta), \eta) = \eta^{1/2}x(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta),$$

(1.8.c)
$$\rho_{\mathrm{can}}(t(\tilde{t}, \eta), \eta) = -\frac{\eta^{1/2}\tilde{\nu}(\tilde{t}, \eta)}{\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta)} - \frac{3\eta^{-1/2}\dfrac{\partial^2 x}{\partial \tilde{x}^2}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta)}{4\left(\dfrac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}(\tilde{t}, \eta), \tilde{t}, \eta)\right)^2},$$

(1.9) $x_{j/2}$ *and* $t_{j/2}$ $(j \geq 2)$ *respectively have the form*

$$\sum_{k=-(j-2)}^{j-2} y_k(\tilde{x}, \tilde{t})e^{k\tilde{\phi}_J(\tilde{t})\eta} \quad \text{and} \quad \sum_{k=-(j-2)}^{j-2} s_k(\tilde{t})e^{k\tilde{\phi}_J(\tilde{t})\eta};$$

that is, $x_{j/2}$ *and* $t_{j/2}$ $(j \geq 2)$ *consist of k-instanton terms with* $|k| \leq j-2$.

Note that (1.8.b) and (1.8.c) implicitly give relations between constants contained in $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$, although they cannot establish a unique correspondence between them.

Actually, after introducing $x(\tilde{x}, \tilde{t}, \eta)$ by that given in Theorem 3.1 of [AKT], we try to construct $t(\tilde{t}, \eta)$ by first requiring

$$(1.10) \qquad \rho_{\mathrm{can}}^2 - 4\sigma_{\mathrm{can}}^2 = \eta \left(\frac{\tilde{\nu}}{\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}, \tilde{t}, \eta)} + \frac{3\eta^{-1}\frac{\partial^2 x}{\partial \tilde{x}^2}(\tilde{\lambda}, \tilde{t}, \eta)}{4(\frac{\partial x}{\partial \tilde{x}}(\tilde{\lambda}, \tilde{t}, \eta))^2} \right)^2 - 4\eta x(\tilde{\lambda}, \tilde{t}, \eta)^2;$$

Surprisingly enough, both sides of (1.10) are independent of $\tilde{t}$, and requiring (1.10) amounts to requiring relations among constants contained in $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$. (See the proof of Lemma 1.1 of [KT2].) The construction of $t(\tilde{t}, \eta)$ is, then, achieved by the induction on j making full use of (1.10). We note that in the course of our argument $t_{j/2}$ (j: an even integer ≥ 2) is determined only modulo an additive constant. This freedom of $t_{j/2}$ is effectively used in our proof of Theorem 2.1. Still more important is the fact that fixing $t_{j/2}$ leads to a unique correspondence between the constants contained in $(\rho_{\mathrm{can}}, \sigma_{\mathrm{can}})$ and $(\tilde{\lambda}, \tilde{\nu})$; this is a key relation for the description of the connection formula for general Painlevé transcendents. (See [AKT], [KT3], and [T] for details.)

Although Proposition 1.1 is concerned with the relation between (SL_J) and (Can), we can further verify the following:

Proposition 1.2. *Let $\psi(x, t, \eta)$ be a WKB solution of (Can) that satisfies (D_{can}) also, and let $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta)$ denote*

$$(1.11) \qquad \left(\frac{\partial x(\tilde{x}, \tilde{t}, \eta)}{\partial \tilde{x}} \right)^{-1/2} \psi(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta).$$

Then $\tilde{\psi}(\tilde{x}, \tilde{t}, \eta)$ satisfies both (SL_J) and (D_J) near the double turning point $\tilde{x} = \tilde{\lambda}_0(\tilde{t})$.

The proof of this proposition is attained by verifying

$$\widetilde{A}_J \frac{\partial x}{\partial \tilde{x}} - \frac{\partial x}{\partial \tilde{t}} - A_{\text{can}} \frac{\partial t}{\partial \tilde{t}} = 0; \tag{1.12}$$

one can readily verify (1.12) guarantees that $\tilde{\psi}$ satisfies not only (SL_J) but also (D_J). (Cf. the proof of Proposition 2.2 of [KT1].)

Remark 2.1. Although $t(\tilde{t}, \eta)$ cannot be uniquely determined by (1.8) when 2-parameters of $(\tilde{\lambda}, \tilde{\nu})$ vanish, $t(\tilde{t}, \eta)$ can be uniquely determined by the limit as the 2-parameters tend to 0. Proposition 1.2 continues to hold for the choice of $t(\tilde{t}, \eta)$ in this degenerate case.

§2. Local equivalence of 2-parameter multiple-scale solutions.

The local equivalence of the simultaneous equations (SL_J) and (D_J) and the simultaneous equations (Can) and (D_{can}) established in the precedent section automatically entails the local equivalence of (SL_J) & (D_J) and (SL_{I}) & (D_{I}) near the double turning point. As one may naturally expect in view of the results in [KT1], this local equivalence can be "matched" with the local equivalence between (SL_J) and (SL_{I}) near the simple turning point (that merges with the double turning point at the turning point for $\tilde{\lambda}_J^{(0)}$ in question). The 'matching' is achieved this time by making use of the freedom in the choice of $t(\tilde{t}, \eta)$ in Proposition 1.1. Once such a semi-global equivalence is constructed, it gives rise to the required local reduction of $\tilde{\lambda}_J$ to λ_{I}. To state the result in a precise manner, let us clarify the geometric situation in which we analyze the problem. (Cf. §2 of [KT1].) Let $\tilde{t}_*$ be a point in a Stokes curve for $\tilde{\lambda}_J^{(0)}$ emanating from a simple turning point $\tilde{r}$ for $\tilde{\lambda}_J^{(0)}$. In what follows, we assume $\tilde{t}_* \neq \tilde{r}$. Then there exist a simple turning point $\tilde{a}(\tilde{t})$ and a Stokes curve $\tilde{\gamma}$ of (SL_J) such that $\tilde{\gamma}$ joins $\tilde{a}(\tilde{t})$ and the double turning point $\tilde{\lambda}_{J,0}(\tilde{t})$. (See Corollary 2.1 of [KT1].) Having this configuration in mind, we obtain the following Theorem 2.1, which is a natural generalization of the local equivalence of 0-parameter Painlevé transcendents (Theorem 2.3 of [KT1]):

Theorem 2.1. *For each 2-parameter formal solution $(\tilde{\lambda}_J, \tilde{\nu}_J)$ of (H_J) that is obtained by multiple-scale analysis ([AKT, §1]), there exists a 2-parameter formal solution $(\lambda_{\mathrm{I}}, \nu_{\mathrm{I}})$ of (H_{I}) for which the following holds: There exist a neighborhood $\widetilde{U}$ of $\tilde{\gamma}$, a neighborhood $\widetilde{V}$ of $\tilde{t}_*$ and holomorphic functions $x_{j/2}(\tilde{x}, \tilde{t}, \eta)$ and $t_{j/2}(\tilde{t}, \eta)$ ($j = 0, 1, 2, \cdots$, $\tilde{x} \in \widetilde{U}$ and $\tilde{t} \in \widetilde{V}$) which satisfy the following:*

(2.1) *The functions x_0 and t_0 are independent of η,*

(2.2.1) $x_0(\tilde{\lambda}_{J,0}(\tilde{t}), \tilde{t}) = \lambda_{\mathrm{I},0}(t_0(\tilde{t}))$,

(2.2.ii) $x_0(\tilde{a}(\tilde{t}), \tilde{t}) = -2\lambda_{\mathrm{I},0}(t_0(\tilde{t}))\ (= a(t_0(\tilde{t})))$,

(2.3) $\dfrac{\partial x_0}{\partial \tilde{x}}$ *never vanishes on* $\widetilde{U} \times \widetilde{V}$,

(2.4) $\tilde{\phi}_J(\tilde{t}) = \phi_{\mathrm{I}}(t_0(\tilde{t}))$,

(2.5) $x_{1/2}$ *and* $t_{1/2}$ *vanish identically,*

(2.6) *Setting* $x(\tilde{x}, \tilde{t}, \eta) = \sum_{j\geq 0} x_{j/2}\eta^{-j/2}$ *and* $t(\tilde{t}, \eta) = \sum_{j \geq 0} t_{j/2}\eta^{-j/2}$, *we find the following:*

(2.6.a) $\widetilde{Q}_J(\tilde{x}, \tilde{t}, \eta) = \left(\dfrac{\partial x}{\partial \tilde{x}}\right)^2 Q_{\mathrm{I}}\,(x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta), \eta) - \frac{1}{2}\eta^{-2}\{x(\tilde{x}, \tilde{t}, \eta); \tilde{x}\}$,

(2.6.b) $x(\tilde{\lambda}_J(\tilde{t}, \eta), \tilde{t}, \eta) = \tilde{\lambda}_{\mathrm{I}}(t(\tilde{t}, \eta), \eta)$.

The relation (2.6.a) implies the transformation $(\tilde{x}, \tilde{t}) \mapsto (x(\tilde{x}, \tilde{t}, \eta), t(\tilde{t}, \eta))$ brings (SL_{I}) into (SL_J), and the transformation gives rise to the required transformation of Painlevé transcendents as is stated in (2.6.b). See [KT2] for the core idea of the proof. The detailed proof will appear in [KT3].

References

[AKT] Aoki, T., T. Kawai and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. II. Structure of Solutions of Differential Equations, World Scientific, 1996, pp. 1-49.

[KT1] Kawai, T. and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. I. Adv. in Math., **118** (1996), 1-33.

[KT2] Kawai, T. and Y. Takei: On the structure of Painlevé transcendents with a large parameter. II. To appear in Proc. Japan Acad. Ser. A.

[KT3] Kawai, T. and Y. Takei: WKB analysis of Painlevé transcendents with a large parameter. III. In prep.

[T] Takei, Y.: On the connection formula for the first Painlevé equation. RIMS-Kôkyûroku, No. 931, pp. 70-99, 1995.